KEY: Assignment Ch 3

3.1 # 2, 18, 32, 36, 40

2) $f(x) = x + \frac{32}{x^2} = x + 32x^{-2}$

$f'(x) = 1 - 64x^{-3} = 1 - \frac{64}{x^3}$

$f'(2) = 1 - \frac{64}{8} = -7$

$f'(4) = 1 - \frac{64}{64} = 0$

$f'(8) = 1 - \frac{64}{512} = \frac{7}{8}$

18) $f(x) = \sqrt{9-x^2} = (9-x^2)^{1/2}$ $\rightarrow$ note: $-3 \le x \le 3$

$f'(x) = \frac{1}{2}(9-x^2)^{-1/2}(-2x)$

$= -\frac{x}{\sqrt{9-x^2}}$

Zero at $x = 0$, undefined at $x = \pm 3$ (critical numbers)

Interval	Sign of $f'(x)$	Conclusion
$(-3, 0)$	$+$	increasing
$(0, 3)$	$-$	decreasing

32) $f(x) = \frac{x^2}{x^2+4}$

$f'(x) = \frac{2x(x^2+4) - 2x(x^2)}{(x^2+4)^2} = \frac{2x^3 + 8x - 2x^3}{(x^2+4)^2} = \frac{8x}{(x^2+4)^2}$

$f'(x) = 0$ when $x = 0$ (critical number)

Interval	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$-$	decreasing
$(0, \infty)$	$+$	increasing

36) $y = \begin{cases} 2x+1 & x \le -1 \\ x^2 - 2 & x > -1 \end{cases}$

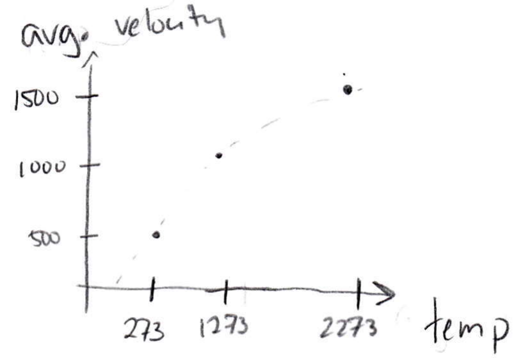
At $x=-1$, for y' 2 and -2 (discontinuity !)

$y' = \begin{cases} 2 & x \le -1 \\ 2x & x > -1 \end{cases}$

$y' = 0$ when $x = 0$ (critical numbers at $x = -1$, $x = 0$)

Interval	Sign of y'	Conclusion
$(-\infty, -1)$	$+$	increasing
$(-1, 0)$	$-$	decreasing
$(0, \infty)$	$+$	increasing

40) Avg. velocity "indicated by peak:"



273k: peak at ≈ 500 m/s

1273k: ≈ 1000 m/s

2273k: ≈ 1500 m/s

273k: increases on (0, 500)
decreases on (500, 1000)

1273k: increase (0, 1000)
decrease (1000, 2000)

2273k: increase (0, 1500)
decrease (1500, 3000)

} endpoints approximate!

3.2 # 4, 6, 16, 30, 38, 48

4) $f(x) = -4x^2 + 4x + 1$

$f'(x) = -8x + 4 \rightarrow f'(x) = 0$ when $x = \frac{1}{2}$

Interval	Sign of $f'(x)$	Conclusion
$(-\infty, \frac{1}{2})$	+	increasing
$(\frac{1}{2}, \infty)$	—	decreasing

there is a rel. maximum at $(\frac{1}{2}, 2)$

6) $g(x) = \frac{1}{5}x^5 - x$

$g'(x) = x^4 - 1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$

$g'(x) = 0$ for $x = \pm 1$

Interval	Sign $f'(x)$	Concl.
$(-\infty, -1)$	+	increase
$(-1, 1)$	—	decrease
$(1, \infty)$	+	increase

relative max. at $(-1, \frac{4}{5})$

relative min. at $(1, -\frac{4}{5})$

16) $f(x) = x + \frac{1}{x} = x + x^{-1}$

$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$ undefined at $x = 0$.
Zero at $x = \pm 1$

Increases $(-\infty, -1)$
decrease $(-1, 0)$ } rel max at $(-1, -2)$

decrease $(0, 1)$
increase $(1, \infty)$ } rel. min. at $(1, 2)$

30) $g(x) = 4\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)$ on $[-4, 5]$

$= 4(1 + x^{-1} + x^{-2})$

$g'(x) = 4(-x^{-2} - 2x^{-3})$ → discontinuity! See graph.

$= -4\left(\frac{1}{x^2} + \frac{2}{x^3}\right)$ undefined for $x=0$ } critical numbers

$= -4\left(\frac{x+2}{x^3}\right)$ zero for $x=-2$

Test	$g(x)$
-4	$13/4 = 3.25$
-2	3
0	undefined $\cancel{x}$
5	$124/25 = 4.96$

At $x=0$, using a limit or graphing utility we find a vertical asymptote such that $g(x) \to \infty$ from the left and the right. ($g(x)$ NOT continuous) So the abs. max. does not exist. The abs. min is at $(-2, 3)$.

See intermediate value theorem.

38) $f(x) = \frac{8}{x+1}$ on $[0, \infty)$ Continuous ✓

$= 8(x+1)^{-1}$

$f'(x) = -8(x+1)^{-2} = -\frac{8}{(x+1)^2}$ Never zero, never undefined on $[0, \infty)$

Test	$f(x)$
0	8

Using a graph, we see that $(0, 8)$ is the absolute maximum.

48) $v = k(R-r)r^2$ $0 \le r < R$.

$= k(Rr^2 - r^3)$

$\frac{dv}{dr} = k(2Rr - 3r^2)$

$= k(2R - 3r)r$

$\frac{dv}{dr} = 0$ when $r=0$ and $2R - 3r = 0$

$2R = 3r$

$\frac{2R}{3} = r$

Test	v
0	0
$\frac{2R}{3}$	$k\left(R - \frac{2R}{3}\right)\left(\frac{2R}{3}\right)^2 = k\left(\frac{3R}{3} - \frac{2R}{3}\right)\frac{4R^2}{9} = k\left(\frac{R}{3}\right)\left(\frac{4R^2}{9}\right) = \frac{4k}{27}R^3$

This will be the maximum velocity as long as $k>0$.

3.3 # 2, 14, 36, 64, 68

2) $y = -x^3 + 3x^2 - 2$

$y' = -3x^2 + 6x$

$y'' = -6x + 6$

$= -6(x-1)$

$y'' = 0$ when $x = 1$

Interval	Sign y''	Conclusion
$(-\infty, 1)$	+	Concave upward
$(1, \infty)$	−	Concave downward

14) $f(x) = x + \frac{4}{x}$

$= x + 4x^{-1}$

$f'(x) = 1 - 4x^{-2}$

$f''(x) = 8x^{-3} = \frac{8}{x^3}$ undefined at $x=0$ never equal to zero.

Interval	Sign f''	Conclusion
$(-\infty, 0)$	−	Concave downward
$(0, \infty)$	+	Concave upward

36) $f(x) = -4x^3 - 8x^2 + 32$

$f'(x) = -12x^2 - 16x$

$f''(x) = -24x - 16$

$f''(x) = 0$ when $-24x = 16$

$x = -\frac{2}{3}$ So this is a point of inflection.

64) $f(x) = -\frac{1}{20}x^5 - \frac{1}{12}x^2 - \frac{1}{3}x + 1$ on $[-2, 2]$

$f'(x) = -\frac{1}{4}x^4 - \frac{1}{6}x - \frac{1}{3}$

$f''(x) = -x^3 - \frac{1}{6}$

Graph and Comment on relationships.

68) $N = -t^3 + 12t^2$, $0 \le t \le 12$

a) $N' = -3t^2 + 24t = -3t(t-8)$ Zero at $t=8$, $t=0$

Test pt	N
0	0
8	256
12	0

The maximum projected # of people is 25,600 at 8 weeks.

b) Most rapid spread will occur when slope is steepest → maximize derivative

$N'' = -6t + 24$ Zero at $t = 4$

Test	N'
0	0
4	48
12	−144

The slope (rate of spread) is maximized at week 4 (4800 people per week)

3.4 # 4, 16, 20, 32, 40

4) First $= x$ Second $= y$, $x, y > 0$

$x + 2y = 100 \rightarrow x = 100 - 2y$

xy is maximum

$(100-2y)y = 100y - 2y^2$ $\quad 0 < y < 50$

$\frac{d}{dy} = 100 - 4y = 0$

$100 = 4y$

$25 = y$ critical pt

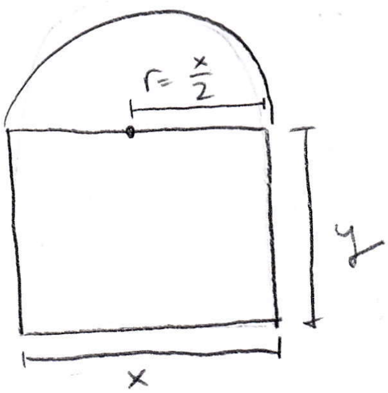
Confirm maximum:

$(0, 25)$ $\quad f'(10) = 60 > 0$ increase

$(25, 50)$ $\quad f'(30) = -20 < 0$ decrease

So $y = 25$, $\quad x = 100 - 50 = 50$

16)



Perimeter is 16 ft. Maximize area.

$A =$ rectangle $+ \frac{1}{2}$ circle

$= xy + \frac{1}{2}\pi r^2$

$= xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = xy + \frac{\pi}{8}x^2$

$0 < x < \frac{32}{2+\pi} \approx 6.22$ $\quad P = x + 2y + \frac{1}{2}(2\pi r)$

$16 = x + 2y + \pi\left(\frac{x}{2}\right)$

$16 = 2y + \left(1 + \frac{\pi}{2}\right)x = 2y + \frac{2+\pi}{2}x$

$8 = y + \frac{2+\pi}{4}x$

$y = 8 - \frac{2+\pi}{4}x = \frac{32 - (2+\pi)x}{4}$

So,

$A = x\left(\frac{32-(2+\pi)x}{4}\right) + \frac{\pi}{8}x^2$

$= 8x - \frac{2+\pi}{4}x^2 + \frac{\pi}{8}x^2 = 8x + \left(-\frac{4+2\pi}{8} + \frac{\pi}{8}\right)x^2$

$= 8x - \left(\frac{4+\pi}{8}\right)x^2$

$A' = 8 - \frac{4+\pi}{4}x = 0 \rightarrow 8 = \frac{4+\pi}{4}x \rightarrow 32 = (4+\pi)x$

$x = \frac{32}{4+\pi} \approx 4.481$

Interval	A'	Conclusion
$(0, 4.48)$	+	increase
$(4.48, 6.22)$	−	decrease

$\Rightarrow$ $x \approx 4.481$, $y \approx 17.923$

20) 16 trees $\rightarrow$ 80 apples / tree

For each additional tree, yield decreases by 4 / tree.

Yield = (# of trees)(apples per tree)

Let t = the number of trees over 16 trees.

then # of trees = $16 + t$

apple / tree = $80 - 4t$

Domain: $16 + t > 0 \rightarrow t > -16$

$80 - 4t > 0$

$80 > 4t$

$t < 20$

$-16 < t < 20$

$$Y = (16+t)(80-4t)$$

$$\frac{dY}{dt} = (16+t)(-4) + (80-4t)(1)$$

$$= -64 - t + 80 - 4t$$

$$= 16 - 5t$$

$$\rightarrow 16 = 5t$$

$$t = \frac{16}{5} = 3.2$$

Interval	Y'	Conclusion
$(-16, 3.2)$	+	increase
$(3.2, 20)$	−	decrease

$\rightarrow$ $t = 3.2$ is a maximum

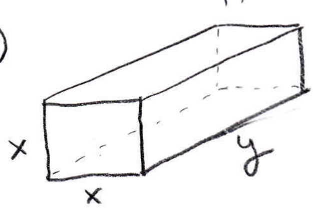
Maximum # of trees = $16 + 3.2 = 19.2$

For 19 trees, $t = 3$ and $Y = (19)(68) = 1292$

For 20 trees, $t = 4$ and $Y = (20)(64) = 1216$

So 19 trees should be planted with a maximum yield of 1292 apples.

32)



length + girth ≤ 108

$y + 4x \leq 108$

Maximize volume: $V = x^2 y$

$y = 108 - 4x$, $\quad 0 < x < \frac{108}{4} = 27$

$$V = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$V' = 216x - 12x^2$$

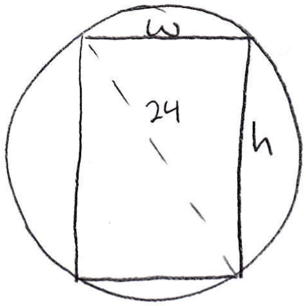
$$= x(216 - 12x)$$

$V' = 0$ when $x = 0$, $\quad x = \frac{216}{12} = 18$

Interval	Conclusion
$(0, 18)$	Increase
$(18, 27)$	Decrease

Maximized at $x = 18$, $\quad y = 36$

40)



$S = k h^2 w$, $k > 0$ constant

$w^2 + h^2 = 24^2$ $w, h > 0$

$h^2 = 24^2 - w^2$

So

$$S = k(24^2 - w^2)w$$
$$= k(24^2 w - w^3)$$
$$S' = k(24^2 - 3w^2)$$
$$0 = k(576 - 3w^2)$$
$$0 = 576 - 3w^2 \qquad k > 0$$
$$w^2 = 192$$
$$w \approx 13.8564$$

And $h^2 = 24^2 - 192$

$= 384$

$h \approx 19.5959$

3.5 # 2, 14, 32, 42, 62

2) $f(x) = \dfrac{4}{(x-2)^3}$ undefined for $x = 2$

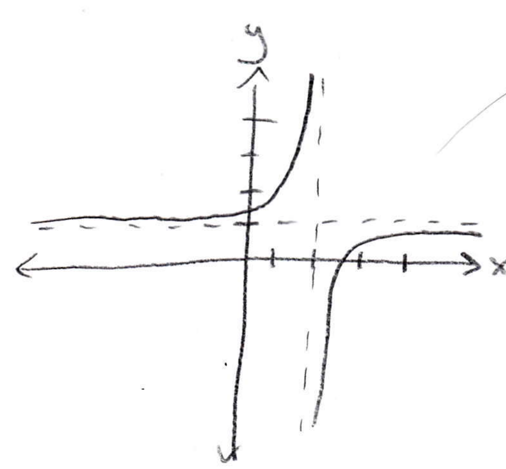
So this is a vertical asymptote

$\left.\begin{array}{l}\deg(\text{numerator}) = 0 \\ \deg(\text{denominator}) = 3\end{array}\right\}$ $y = 0$ is a horizontal asymptote

14) $\displaystyle\lim_{x \to -2^-} \left(\frac{1}{x+2}\right) = -\infty$

32) $\displaystyle\lim_{x \to \infty} \frac{5x^3 + 1}{10x^3 - 3x^2 + 7} = \frac{5}{10} = 0.5$

42) $y = \dfrac{x-3}{x-2}$ asymptotes: $y = 1$ $x = 2$



62) $C = \frac{3t-1}{2t^2+5}$, $t > 0$

a) Using Desmos, maybe around $t=2$.

b) deg(denom) > deg(num), so horiz. asymptote at $y=0$

Essentially, medication concentration will eventually reach (nearly) zero.