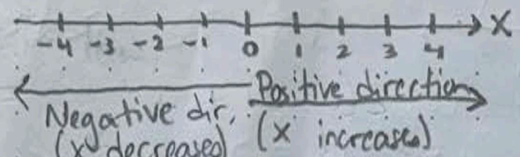
Chapter 0 - Review of Algebra/Precalculus

~~Arithmetic Operations~~

0.1 The Real Number Line and Order

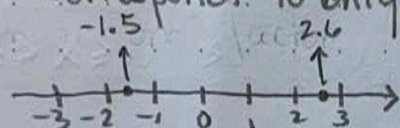
> The Real Number Line

We can represent real numbers using the <u>real number line</u> (think x-axis)

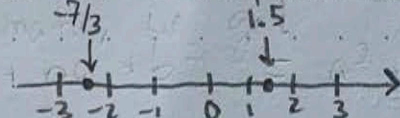


<u>Coordinate</u>: the number corresponding to a point on the real number line.
<u>Origin</u>: the point corresponding to zero.
Points to the right of the origin are <u>positive</u>.
Points to the left are <u>negative</u>.
The term <u>nonnegative</u> refers to all positive numbers AND zero.

Each point corresponds to only one real number.



Each real number corresponds to only one point.



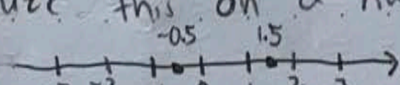
We call this a <u>one-to-one correspondence</u>.

If a number can be represented as the ratio of two integers $\frac{7}{3}$, $-\frac{3}{2}$, etc.
we call it <u>rational</u>.

If we are unable to represent a number this way, we say that it is <u>irrational</u>. <u>Ex</u> π, $\sqrt{2}$, e

> Order and Intervals on the Real Number Line

Real numbers are <u>ordered</u> - we can sort any group of real numbers.
We can visualize this on a number line



Notice that -0.5 is to the left of 1.5. This means that -0.5 is less than 1.5. We write

$$-0.5 < 1.5$$

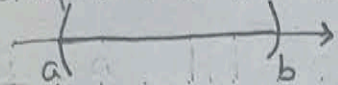
less than

We can also order more than two numbers. If $a<x$ and $x<b$,

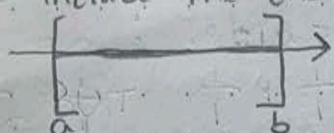
$$a < x < b$$

x is <u>between</u> a and b.

The set of real numbers b/w a and b is an <u>open interval</u> which we denote (a,b). It is "open" b/c it does not include the endpoints a and b.

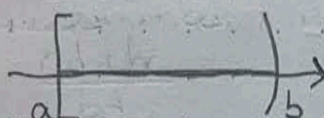

(a,b) OR $a<x<b$

<u>Closed intervals</u> include the endpoints.


$[a,b]$ OR $a \le x \le b$

Some intervals are neither open nor closed:

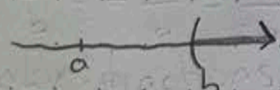

$(a,b]$ OR $a<x\le b$

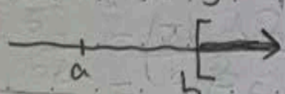

$[a,b)$ OR $a \le x < b$

There are also infinite intervals:


$(-\infty, a)$ OR $x<a$


$(-\infty, a]$ OR $x \le a$


(b, ∞) OR $x>b$


$[b,\infty)$ OR $x \ge b$


$(-\infty, \infty)$ OR all real numbers!

> Solving Inequalities

Properties of Inequalities: for a, b, c, d real numbers,

1) $a<b$ and $b<c \Rightarrow a<c$ — Transitive property
2) $a<b$ and $c<d \Rightarrow a+c < b+d$ — Adding inequalities
3) Multiply by positive constant: $a<b \Rightarrow ac<bc$, $c>0$
4) Multiply by negative constant: $a<b \Rightarrow ac>bc$, $c<0$
5) Adding a constant: $a<b \Rightarrow a+c < b+c$

   $a<b \Rightarrow a-c < b-c$

This is a lot like solving equalities! The big difference is #4 where we need to flip the equality.

Ex $3 < 5$

Multiplying by -2, $\quad -2(3) > -2(5)$

$$-6 > -10$$

Ex Find the solution set for $3x - 4 < 5$

$$3x - 4 < 5$$
$$3x - 4 + 4 < 5 + 4$$
$$3x < 9$$
$$\tfrac{1}{3}(3x) < \tfrac{1}{3}(9)$$
$$x < 3$$

The soln set is the interval $(-\infty, 3)$

Checkpoint Find the soln set for the inequality $2x - 3 < 7$.

Solving polynomial inequalities

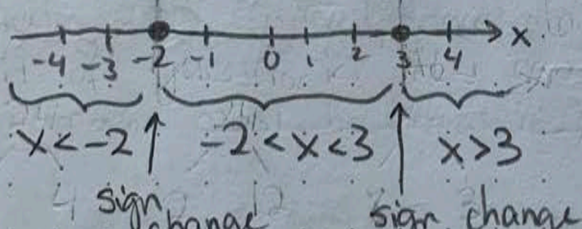
Recall that polynomials change sign only at numbers that make the polynomial zero. We will use this for polynomial inequalities.

Ex Consider $x^2 < x + 6$

$x^2 - x - 6 < 0$ move everything to one side

$(x-3)(x+2) < 0$ factor

This has zeros at $x = 3$ and $x = -2$.



Choose a number in each interval to plug into the inequality

x	Sign	<0
-3	+	No
0	−	Yes
4	+	No

So the soln set is $(-2, 3)$

Checkpoint Find the soln set for $x^2 > 3x + 10$

0.2 Absolute Value and Distance on the Real Number Line

Def The absolute value of a real number c is

$$|c| = \begin{cases} c & \text{if } c \geq 0 \\ -c & \text{if } c < 0 \end{cases}$$

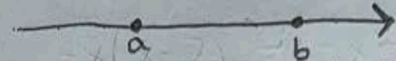
ex $|-3| = -(-3) = 3$

Properties:

1) Multiplication: $|ab| = |a||b|$
2) Division: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$
3) Power: $|a^n| = |a|^n$
4) Square root: $\sqrt{a^2} = |a|$

ex if $a = -2$, $\sqrt{(-2)^2} = \sqrt{4} = 2 = |-2|$

Distance on the Real Number Line



The directed distance from a to b is $\xrightarrow{b-a}$

The directed distance from b to a is $\xleftarrow{a-b}$

The distance between a and b is $|a-b|$ OR $|b-a|$

We can also write $\sqrt{(a-b)^2}$

Ex Determine the distance between -3 and 4. What is the directed distance from -3 to 4? from 4 to -3?

$$d = |4-(-3)| = |7| = 7$$

directed distance -3 to 4 is $4-(-3) = 7$
" 4 to -3 is $-3-4 = -7$

Intervals Defined by Absolute Value

Ex Find the interval that contains all real numbers that lie no more than two units from 3.

Soln
$|x-3| \leq 2$ So $x-3$ is b/w -2 and 2

$-2 \leq x-3 \leq 2$

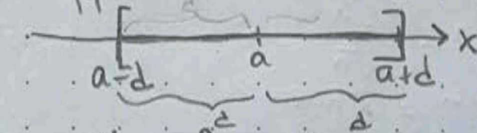
$1 \leq x \leq 5$

$[1, 5]$

Two types of absolute value inequalities:
Let a, d be real numbers, $d > 0$

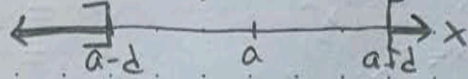
1) $|x-a| \le d$ iff $-d \le x-a \le d$
2) $|x-a| \ge d$ iff $x-a \le -d$ OR $d \le x-a$

$|x-a| \le d$



"x is <u>within</u> d units of a"

$|x-a| \ge d$



"x is <u>at least</u> d units from a"

Midpoint of an interval:
The <u>midpoint</u> of an interval is found by taking the average of the two endpoints. For an interval with endpoints a and b,

$$\text{Midpoint} = \frac{a+b}{2}$$

<u>0.3 Exponents and Radicals</u>

Properties of Exponents:

1) Whole number exponents: $x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$
2) Zero exponent: $x^0 = 1$, $x \neq 0$
3) Negative exponents: $x^{-n} = \frac{1}{x^n}$, $x \neq 0$
4) Radicals (principal n^{th} root): $\sqrt[n]{x} = a \Rightarrow x = a^n$
5) Rational exponents $(\frac{1}{n})$: $x^{1/n} = \sqrt[n]{x}$
6) Rational exponents $(\frac{m}{n})$: $x^{m/n} = (x^{1/n})^m = (\sqrt[n]{x})^m$
$x^{m/n} = (x^m)^{1/n} = \sqrt[n]{x^m}$
7) Special convention (square root): $\sqrt[2]{x} = \sqrt{x}$

<u>ex</u> $y = 3x^{-3}$ for $x = -1$

<u>Soln</u>
$$y = 3(-1)^{-3} = \frac{3}{(-1)^3} = \frac{3}{-1} = -3$$

<u>ex</u> $y = \sqrt[3]{x^2}$ for $x = 8$

<u>Soln</u>
$$y = \sqrt[3]{8^2} = 8^{2/3} = (8^{1/3})^2 = 2^2 = 4$$

Operations with Exponents:

1) Multiplying like bases: $x^n x^m = x^{n+m}$

2) Dividing like bases: $\frac{x^n}{x^m} = x^{n-m}$

3) Removing parentheses: $(xy)^n = x^n y^n$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$(x^n)^m = x^{nm}$$

4) Special conventions: $-x^n = -(x^n)$, $\quad -x^n \neq (-x)^n$

$$cx^n = c(x^n) \qquad cx^n \neq (cx)^n$$

$$x^{n^m} = x^{(n^m)} \qquad x^{n^m} \neq (x^n)^m$$

ex Simplify $2x^2(x^3)$

Soln $2x^2(x^3) = 2x^2x^3$

$$= 2x^{2+3}$$

$$= 2x^5$$

ex Simplify $\frac{3x^2}{(x^{1/2})^3}$

Soln

$$\frac{3x^2}{(x^{1/2})^3} = \frac{3x^2}{x^{3/2}}$$

$$= 3x^{2-3/2}$$

$$= 3x^{1/2}$$

$$= 3\sqrt{x}$$

Distributive property:

$$abx^n + acx^{n+m} = ax^n(b + cx^m)$$

ex Simplify $2x^3 + x^2$

Soln $2x^3 + x^2 = 2x^{2+1} + x^2$

$$= 2x^2x^1 + x^2$$

$$= x^2(2x+1)$$

ex Simplify $2x^{-1/2} + 3x^{5/2}$

Soln $2x^{-1/2} + 3x^{5/2}$

$$= 2x^{-1/2} + 3x^{6/2 - \frac{1}{2}}$$

$$= 2x^{-1/2} + 3x^{6/2}x^{-1/2}$$

$$= x^{-1/2}(2 + 3x^3)$$

$$\left\langle = \frac{2+3x^2}{\sqrt{x}} \right\rangle$$

Sometimes in calculus we will need to simplify messier expressions:

ex $3(x+1)^{1/2}(2x-3)^{5/2} + 10(x+1)^{3/2}(2x-3)^{3/2}$

$$= 3(x+1)^{1/2}(2x-3)^{3/2+2/2} + 10(x+1)^{1/2+2/2}(2x-3)^{3/2}$$

$$= (x+1)^{1/2}(2x-3)^{3/2}[3(2x-3) + 10(x+1)]$$

$$= (x+1)^{1/2}(2x-3)^{3/2}(6x-9+10x+10)$$

$$= (x+1)^{1/2}(2x-3)^{3/2}(16x+1)$$

ex $\frac{3x^2 + x^4}{2x} = \frac{x^2(3+x^2)}{2x} = \frac{x(3+x^2)}{2}$

Domain of an Algebraic Expression:

Def: The set of all numbers for which the expression is defined is called its domain.

ex Find the domain of $(3x-2)^{-1/2}$

Soln $\frac{1}{\sqrt{3x-2}}$

First, note this is undefined if $\sqrt{3x-2} = 0$
It is also undefined if $3x-2 < 0$

So the domain must be $3x-2 > 0$ (positive and nonzero)
$3x > 2$
$x > 2/3$

And the domain is $(2/3, \infty)$

0.4 Factoring Polynomials

The Fundamental Theorem of Algebra states that every n^{th} degree polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 x^0, \quad a_n \neq 0$$

has precisely n zeros (which may be repeated or imaginary)

Special Products and Factorization Techniques:

Quadratic Formula:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Special Products:

$$x^2 - a^2 = (x-a)(x+a)$$
$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$
$$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$$
$$x^4 - a^4 = (x-a)(x+a)(x^2 + a^2)$$

Binomial Theorem

$$(x+a)^2 = x^2 + 2ax + a^2$$
$$(x-a)^2 = x^2 - 2ax + a^2$$
$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$
$$(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$$

$(x+a)^n$

$$= x^n + nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} + \frac{n(n-1)(n-2)}{3!}a^3x^{n-3} + \dots + na^{n-1}x + a^n$$

$(x-a)^n$

$$= x^n - nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} - \frac{n(n-1)(n-2)}{3!}a^3x^{n-3} + \dots \pm na^{n-1}x \mp a^n$$

Factoring by Grouping

$$acx^3 + adx^2 + bcx + bd = ax^2(cx+d) + b(cx+d)$$
$$= (ax^2+b)(cx+d)$$

<u>Ex</u> Find the zeros of $4x^2 + 6x + 1$

Soln: Quadratic formula. $a=4$ $b=6$ $c=1$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4\times4\times1}}{2\times4}$$
$$= \frac{-6 \pm \sqrt{36-16}}{8}$$
$$= \frac{-6 \pm \sqrt{20}}{8}$$
$$= \frac{-6 \pm 2\sqrt{5}}{8}$$
$$= \frac{-3 \pm \sqrt{5}}{4}$$

$\frac{-3-\sqrt{5}}{4} \approx -1.309$ and $\frac{-3+\sqrt{5}}{4} \approx -0.191$

<u>Ex</u> Find the zeros of $2x^2 - 6x + 5$

<u>Soln</u>: $a=2$ $b=-6$ $c=5$

$$x = \frac{6 \pm \sqrt{36-40}}{4} = \frac{6 \pm \sqrt{-4}}{4}$$

$\boxed{8}$ now $\sqrt{-4}$ is imaginary, so there are no real zeros.

Ex Find the domain of $\sqrt{x^2-3x+2}$

Soln The expression is defined for $x^2-3x+2 \geq 0$.
Find the zeros to find where the sign changes:

$$x^2-3x+2 = (x-1)(x-2)$$

$x=1$ and $x=2$

Setting	Plug in	x^2-3x+2	$\sqrt{x^2-3x+2}$
$x<1$	0	2	$\sqrt{2}$
$1<x<2$	1.5	-0.25	Undefined
$x>2$	3	2	$\sqrt{2}$

So the domain is $(-\infty, 1] \cup [2, \infty)$
$\wedge$ denotes the union - basically an "and"!

Synthetic Division for a Cubic Polynomial

If $x=x_1$ is a zero of ax^3+bx^2+cx+d,

$$\begin{array}{c|cccc} x_1 & a & b & c & d \\ & & \bigcirc & \bigcirc & \bigcirc \\ \hline & a & s & t & \bigcirc \end{array}$$

Vertical pattern: add terms
Diagonal pattern: multiply by x_1

should get zero!

$(x-x_1)(ax^2+sx+t)$

ex Synthetic division on x^3-4x^2+5x-2 with zero $x=2$

$$\begin{array}{c|cccc} 2 & 1 & -4 & 5 & -2 \\ \hline & & & & \end{array}$$

$$\begin{array}{c|cccc} 2 & 1 & -4 & 5 & -2 \\ & & 2 & -4 & 2 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$(x-2)(x^2-2x+1)$

1) Add 1
2) $1 \times 2 = 2$
3) $-4+2=-2$
4) $-2 \times 2 = -4$
5) $5-4=1$
6) $1 \times 2 = 2$
7) $-2+2=0$

Rational Zero Theorem

If a polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

has integer coefficients, then every rational zero is of the form $x=p/q$ where p is a factor of a_0 and q is a factor of a_n.

ex Find all real zeros of $2x^3 + 3x^2 - 8x + 3$

Soln ① rational zero theorem: $a_0 = 3$, has factors ± 1 and ± 3 (p's)
$a_n = 2$, has factors ± 1 and ± 2 (q's)

possible rational zeros p/q are
$1, -1, 3, -3, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}$

plug in 1:
$2(1)^3 + 3(1)^2 - 8(1) + 3 = 0$

② We know 1 is a zero, so we can perform synthetic division for $x-1$:

$$\begin{array}{c|cccc} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$(x-1)(2x^2 + 5x - 3)$

③ Factor the quadratic:
$(x-1)(2x-1)(x+3)$

So the zeros are $x=1$, $x=\frac{1}{2}$, and $x=-3$

0.5 Fractions and Rationalization

Operations with Fractions

1) Add and subtract fractions (find a common denominator)

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\left(\frac{d}{d}\right) + \frac{c}{d}\left(\frac{b}{b}\right) = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b}\left(\frac{d}{d}\right) - \frac{c}{d}\left(\frac{b}{b}\right) = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd}$$

for $b, d \neq 0$.

2) Multiply fractions

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd} \quad ; \quad b, d \neq 0$$

3) Divide fractions (invert and multiply)

$$\frac{a/b}{c/d} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$$

4) Divide out like factors

$$\frac{ab}{ac} = \frac{b}{c}, \quad \frac{ab+ac}{ad} = \frac{a(b+c)}{ad} = \frac{b+c}{d}$$

$a, c, d \neq 0$

ex Simplify $\frac{1}{x+1} - \frac{2}{2x-1}$

Soln

$$\frac{1}{x+1} - \frac{2}{2x-1} = \frac{1}{x+1}\left(\frac{2x-1}{2x-1}\right) - \frac{2}{2x-1}\left(\frac{x+1}{x+1}\right)$$

$$= \frac{2x-1}{(x+1)(2x-1)} - \frac{2(x+1)}{(2x-1)(x+1)}$$

$$= \frac{2x-1-2x-2}{2x^2+x-1}$$

$$= \frac{-3}{2x^2+x-1}$$

ex Simplify $\frac{1}{2(x^2+2x)} - \frac{1}{4x}$

Soln

$$\frac{1}{2(x^2+2x)} - \frac{1}{4x} = \frac{1}{2x(x+2)} - \frac{1}{2(2x)}$$

$$= \frac{1}{2x(x+2)}\left(\frac{2}{2}\right) - \frac{1}{2(2x)}\left(\frac{x+2}{x+2}\right)$$

$$= \frac{2}{4x(x+2)} - \frac{x+2}{4x(x+2)}$$

$$= \frac{2-x-2}{4x(x+2)}$$

$$= \frac{-\cancel{x}}{4\cancel{x}(x+2)}$$

$$= \frac{-1}{4(x+2)}$$

Note: To add more than 2 fractions, find a common denominator among all the fractions. (You can also add 2 at a time if you prefer!)

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15}{30} + \frac{10}{30} + \frac{6}{30} = \frac{31}{30}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{3}{6} + \frac{2}{6} + \frac{1}{5} = \frac{5}{6} + \frac{1}{5} = \frac{25}{30} + \frac{6}{30} = \frac{31}{30}$$

ex Simplify $\frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{x+4}$

Soln. The least common denominator is $(x+2)(x-3)(x+4)$

$$\frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{x+4}$$
$$= \frac{A(x-3)(x+4) + B(x+2)(x+4) + C(x+2)(x-3)}{(x+2)(x-3)(x+4)}$$
$$= \frac{A(x^2+x-12) + B(x^2+6x+8) + C(x^2-x-6)}{(x+2)(x-3)(x+4)}$$
$$= \frac{Ax^2 + Ax - 12A + Bx^2 + 6Bx + 8B + Cx^2 - Cx - 6C}{(x+2)(x-3)(x+4)}$$

Rationalization Techniques

1) If the denominator is $\sqrt{a}$, multiply by $\frac{\sqrt{a}}{\sqrt{a}}$

2) If the denominator is $\sqrt{a} - \sqrt{b}$, multiply by $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

3) If the denominator is $\sqrt{a} + \sqrt{b}$, multiply by $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

The same guidelines apply to rationalizing numerators.

ex Rationalize the denominator for $\frac{1}{\sqrt{x}-\sqrt{x+1}}$

$$\frac{1}{\sqrt{x}-\sqrt{x+1}} = \frac{1}{\sqrt{x}-\sqrt{x+1}}\left(\frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}}\right)$$
$$= \frac{\sqrt{x}+\sqrt{x+1}}{x-(x+1)} = \frac{\sqrt{x}+\sqrt{x+1}}{-1}$$
$$= -\sqrt{x} - \sqrt{x+1}$$