1.2 Graphs of Equations

Learning Outcomes:
- Sketch graphs of equations
- Find the x- and y-intercepts of graphs of equations
- Write the standard forms of equations of circles
- Find the points of intersection of two curves
- Model and solve real-world problems.

> The graph of an equation.

Often, a relationship between two quantities is expressed as an equation.

For example, we convert Celcuis to Fahrenheit using

$$F = \frac{9}{5}C + 32.$$

The graph of an equation is the set of all points that are solutions to the equation.

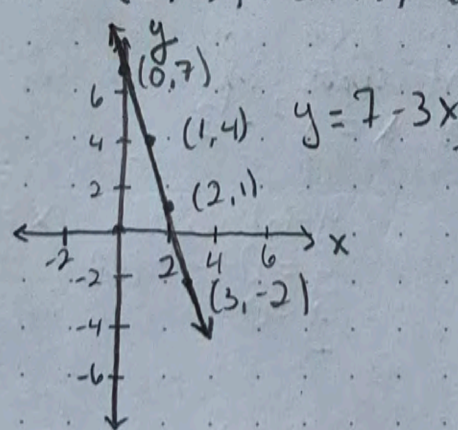
Ex Sketch the graph of $y = 7 - 3x$.

Soln The simplest way to graph an equation is by plotting several points.

$y = 7 - 3x$

x	0	1	2	3
y	7	4	1	-2

This gives us $(0,7)$, $(1,4)$, $(2,1)$, $(3,-2)$



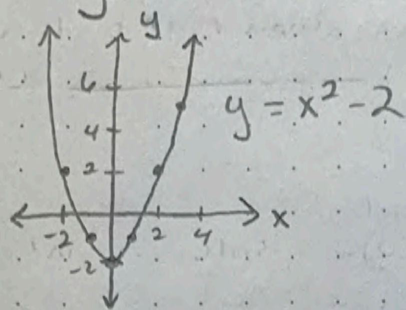
The graph extends as far as you draw it - this line goes on forever!

If you remember your slope-intercept method for plotting, that will also work! But we want to be able to plot more complex equations like

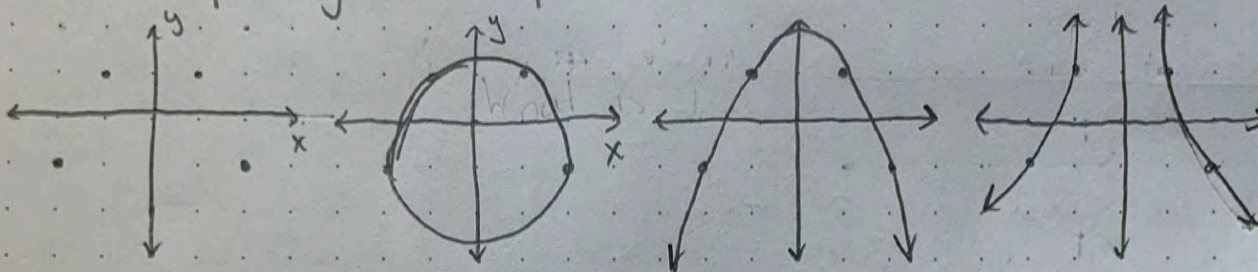
Ex Sketch the graph of $y = x^2 - 2$

Soln

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7



Point plotting is easy but can cause issues.



> Intercepts of a Graph

<u>Intercepts</u> are points at which the graph intersects an axis.

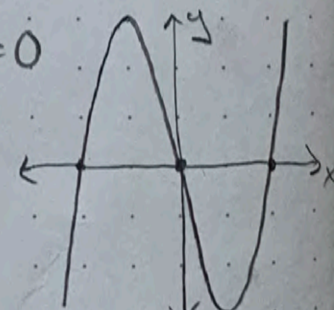
- To find the x-intercepts, let $y=0$ and solve for x.
- To find the y-intercepts, let $x=0$ and solve for y.

<u>Ex</u> Find the x- and y-intercepts of
a) $y = x^3 - 4x$ b) $x = y^2 - 3$

<u>Soln</u> a) x-intercept: $y=0$

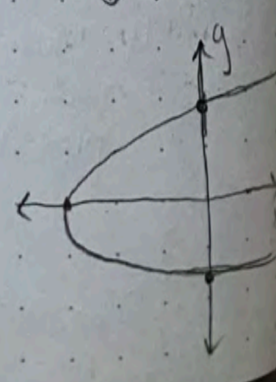
$0 = x^3 - 4x$
$= x(x^2-4)$
$= x(x+2)(x-2)$
$x = 0,\ x = \pm 2$
$(0,0), (2,0), (-2,0)$

y-intercept $x=0$
$y = 0^3 - 4(0)$
$y = 0$
$(0,0)$



b) $x = 0^2 - 3$
$x = -3$
$(-3,0)$

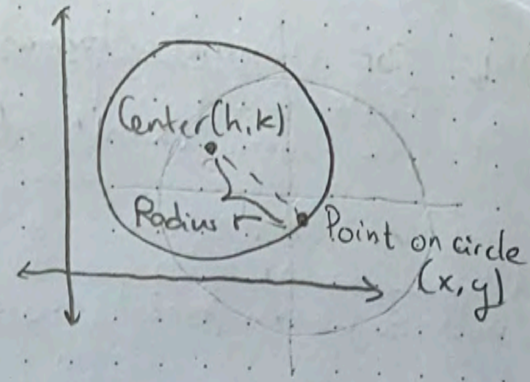
$0 = y^2 - 3$
$= (y+\sqrt{3})(y-\sqrt{3})$
$y = \sqrt{3},\ y = -\sqrt{3}$
$(0,\sqrt{3}), (0,-\sqrt{3})$



> Circles

By the distance formula,

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$



We can use this to get the standard eq'n for a circle:

The point (x, y) lies on the circle of radius r and center (h, k) iff

$$r^2 = (x-h)^2 + (y-k)^2$$

A circle centered at the origin $(0,0)$ is

$$r^2 = x^2 + y^2$$

Ex The point $(3,4)$ lies on the circle whose center is at $(-1, 2)$. Find the equation for this circle.

Sol'n $r = \sqrt{[3-(-1)]^2 + (4-2)^2} = \sqrt{16+4} = \sqrt{20}$

Then $(\sqrt{20})^2 = [x-(-1)]^2 + (y-2)^2$

$$20 = (x+1)^2 + (y-2)^2$$

General form for a circle

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \quad A \neq 0$$

To convert back to the standard form, we must complete the square.

Ex $4x^2 + 4y^2 + 20x - 16y + 37 = 0$

1) Divide by 4 to make x^2, y^2 coef. 1
2) Group terms
3) Complete square

$$x^2 + y^2 + 5x - 4y + \frac{37}{4} = 0$$

$$(x^2 + 5x + \_\_) + (y^2 - 4y + \_\_) = \frac{-37}{4}$$

$$(\underbrace{x^2 + 5x + (\tfrac{5}{2})^2}_{(\text{half})^2}) + (\underbrace{y^2 - 4y + (-2)^2}_{(\text{half})^2}) = -\frac{37}{4} + \frac{25}{4} + 4$$

$$(x + \tfrac{5}{2})^2 + (y-2)^2 = 1$$

Note: The general form may not always result in a circle, e.g. if $(x-h)^2 + (y-k)^2 =$ negative number.

> Points of Intersection

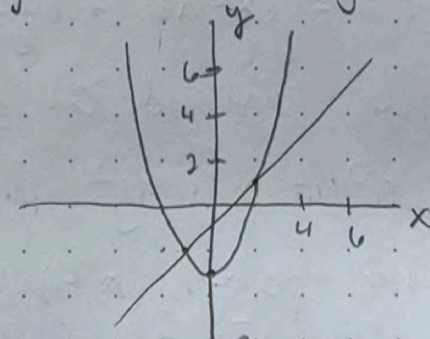
A point of intersection of two graphs is an ordered pair that is a solution point of both graphs.

To find the points of intersection, solve each equation for y and set the y-values equal to each other. Solve.

Ex Find the points of intersection of $y = x^2 - 3$ and $y = x - 1$.

Soln

$$\left.\begin{aligned} x^2 - 3 &= x - 1 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) & \\ x = 2,\ x &= -1 \end{aligned}\right\} \Rightarrow \begin{matrix} (2, 1) \\ (-1, -2) \end{matrix}$$



Ex From 2000 through 2006, the state population of Arizona and the state population of Tennessee could be approximated by

$$P = 5129 + 164.4t \qquad \text{(P in thousands)}$$

and

$$P = 5687 + 54.4t,$$

respectively, where t is the year and $t = 0$ is 2000. When do you expect the two states to have the same population?

Soln

$$5129 + 164.4t = 5687 + 54.4t$$
$$5129 + 110t = 5687$$
$$110t = 558$$
$$t \approx 5.07$$

Or sometime in 2005.

> Mathematical Models

A mathematical model uses equations to represent real-life phenomena. Ideally, these are both accurate and simple, but these goals often conflict.

Ex.

Year	2000	2001	2002	2003	2004	2005
t	0	1	2	3	4	5
Crude Oil	12.358	12.282	12.163	12.026	11.503	10.963

Table: Crude oil energy (in quadrillion Btu) produced in the United States in the years 2000-2005.

Projected 2006 energy production from crude oil was 10.9 quad. Btu. How did they obtain this projection?

Soln: Past production was used to predict future production. The equation was found using a statistical procedure called least squares regression analysis.

$$E = -0.0703t^2 + 0.0815t + 12.323$$

For 2006, $t=6$ so

$$E = -0.0703(6^2) + 0.0815(6) + 12.323$$
$$= 10.282$$

which is similar to the number predicted by the US Energy Information Assoc.

Note: To examine model accuracy, we can compare the models predicted values to the true data values.

Model	12.323	12.334	12.205	11.935	11.524	10.973

Our study of calculus will focus on model behavior. Familiarity with the following 6 graphs will be helpful.