1.4 Functions

Learning outcomes:

- Decide whether relations between two variables are functions.
- Find the domain and range of functions.
- Use function notation and evaluate functions.
- Combine functions to create other functions.
- Find inverse functions algebraically.

Functions describe relationships between variables. Consider the area of a circle

$$A = \pi r^2$$

The area A depends on the value of r. We call A the dependent variable and r the independent variable.

Def A function is a relationship between two variables such that for each value of the independent variable there exists exactly one value of the dependent variable.

Def The domain of a function is the set of all values of the independent variable for which the function is defined.

Def The range of a function is the set of all value taken on by the dependent variable.

A function

Domain, x (x is independent variable)
↓
Input value
↓
Function happens → output
↓
range, y (y is dependent variable)

ex. Does the eqn define y as a function of x?

a) $x+y=1$

$y = 1-x$

Yes.

b) $x^2+y^2=1$

$y^2 = 1-x^2$

$y = \pm\sqrt{1-x^2}$

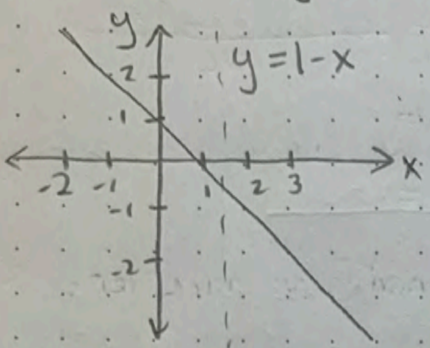
No! Choose $x=0$

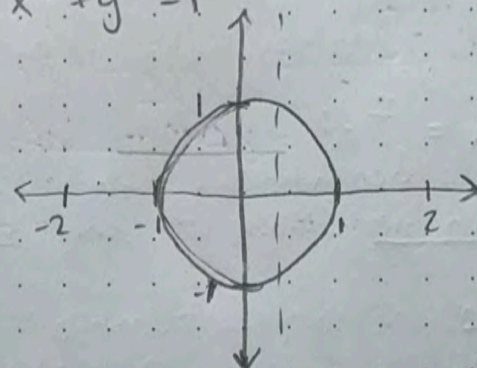
$y = \pm 1$

The Graph of a Function

Convention: When graphing a function, the horizontal axis represents the independent variable.
In this case, we can use the <u>vertical line test</u>:

<u>Ex</u> a) $x + y = 1$ b) $x^2 + y^2 = 1$





"y as a function of x" means y is dependent on x.

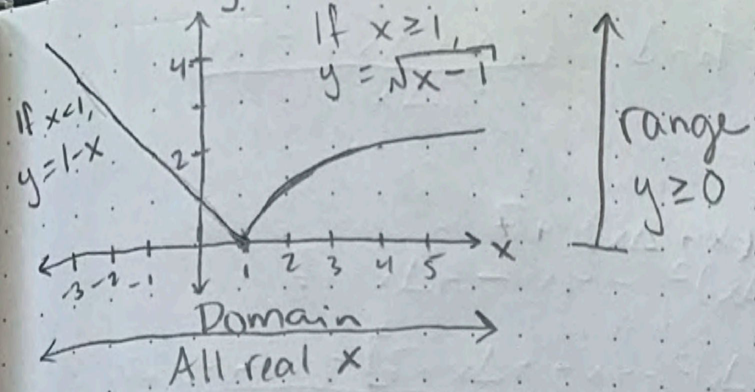
The <u>vertical line test</u> states that if every vertical line intersects the graph of an equation no more than once, then y is a function of x.

<u>Ex</u> Find the domain and range of each function.
a) $y = 1 - x$
b) $y = \sqrt{x-1}$
c) $y = \begin{cases} 1-x & x < 1 \\ \sqrt{x-1} & x \geq 1 \end{cases}$

a) $y = 1 - x$ can take any value and output any value
<u>domain</u>: all reals
<u>range</u>: all reals

b) $y = \sqrt{x-1}$ the negative of a square root is not defined
So $x - 1 \geq 0 \rightarrow x \geq 1$
<u>domain</u>: $x \geq 1$
Then y can take any nonnegative value
<u>range</u>: $y \geq 0$

c) $y = \begin{cases} 1-x & x < 1 \\ \sqrt{x-1} & x \geq 1 \end{cases}$ defined for all x
<u>domain</u> all reals
For $1-x$, $x < 1$ y will always be positive
For $\sqrt{x-1}$, $x \geq 1$, y "
<u>range</u>: $y \geq 0$



Def A function is <u>one-to-one</u> if to each value of the dependent variable in the range, there is exactly one value of the independent variable.

A function is one-to-one if it passes a <u>horizontal line test</u>, where each horizontal line intersects the graph at most once.

A one-to-one function must satisfy both the vertical and horizontal line tests.

> Function notation

With the dependent variable isolated, we can write

$y = 3x - 4$ as $f(x) = 3x - 4$

"y as a function of x"

the function of x
"f of x"

If we wanted $f(x) = 3x - 4$ when $x = 2$, we would write

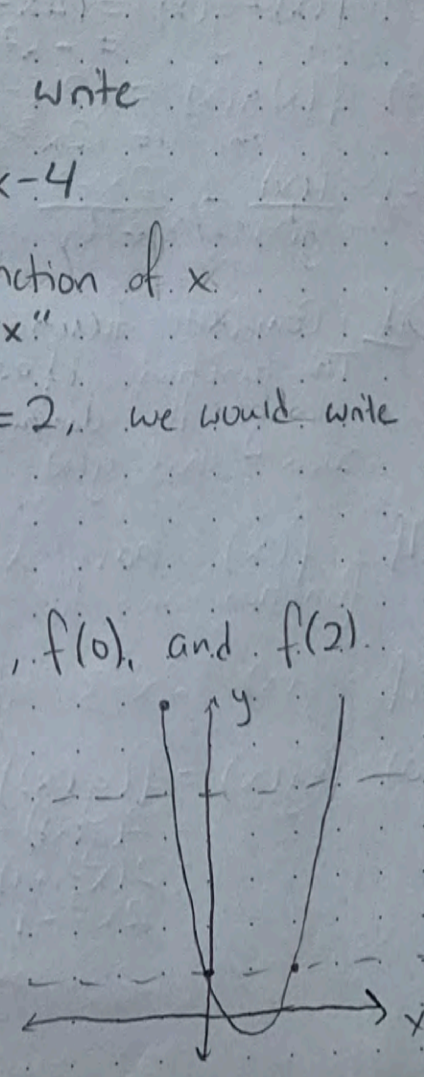
$f(2) = 3 \times 2 - 4 = 2$

function value

Ex $f(x) = 2x^2 - 4x + 1$. Find $f(-1)$, $f(0)$, and $f(2)$. Is f one-to-one?

Soln

$$\left.\begin{aligned} f(-1) &= 2 + 4 + 1 = 7 \\ f(0) &= 1 \\ f(2) &= 8 - 8 + 1 = 1 \end{aligned}\right\} \underline{\text{Not}} \text{ one-to-one!}$$



Ex Let $f(x) = x^2 - 4x + 7$. Find $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$$f(x) = x^2 - 4x + 7$$
$$f(x + \Delta x) = (x + \Delta x)^2 - 4(x + \Delta x) + 7$$
$$= x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 7$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{[x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 7] - (x^2 - 4x + 7)}{\Delta x}$$
$$= \frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x}$$
$$= 2x + \Delta x - 4, \quad \Delta x \neq 0$$

> Combinations of Functions

If $f(x) = 2x - 3$ and $g(x) = x^2 + 1$,

1) $f(x) + g(x) = (2x-3) + (x^2+1)$ — Add
$= x^2 + 2x - 2$
2) $f(x) - g(x) = (2x-3) - (x^2+1)$ — Subtract
$= -x^2 + 2x - 4$
3) $f(x)g(x) = (2x-3)(x^2+1)$ — Multiply
$= 2x^3 - 3x^2 + 2x - 3$
4) $\dfrac{f(x)}{g(x)} = \dfrac{2x-3}{x^2+1}$ — Divide

Def Composite functions:
The function $(f \circ g)(x) = f(g(x))$ is the Composite of f with g. Its domain is the set of x in the domain of g such that $g(x)$ is in the domain of f.

< If $f(x)$ has x as a placeholder, we can plug another function in for x. >

ex If $f(x) = 2x - 3$ and $g(x) = x^2 + 1$

a) $f \circ g(x) = f(g(x))$
$= 2(g(x)) - 3$
$= 2(x^2 + 1) - 3$
$= 2x^2 - 1$

b) $g \circ f(x) = g(f(x))$
$= (f(x))^2 + 1$
$= (2x-3)^2 + 1$
$= 4x^2 - 12x + 10$

If $f(x)$ is "f of x" think of $f \circ g(x)$ as "f of $g(x)$".

> Inverse functions

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x)) = x$$

An inverse function "undoes" whatever f does.

Def. Let f and g be two functions such that

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g.$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.$$

Then the function g is the <u>inverse function</u> of f. We denote this by f^{-1} ("f inverse"), so

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

Note: The domain of f must equal the range of f^{-1}.
The range of f must equal the domain of f^{-1}.

ex Find the inverse of $f(x) = \sqrt{2x-3}$

The inverse gives us back x, so we need to solve for x.

$y = \sqrt{2x-3}$ replace $f(x)$ with y

$y^2 = 2x-3$

$\frac{y^2+3}{2} = x$

This is a function $g(y)$. The inverse function $f^{-1}(x)$ then takes the form

$$f^{-1}(x) = \frac{x^2+3}{2}$$

Fact: For a function to have an inverse function, it must be one-to-one.

Ex $f(x) = x^2 - 1$ OR $y = x^2 - 1$

$f(2) = 3$ $\quad$ $y+1 = x^2$

$f(-2) = 3$ $\quad$ $\pm\sqrt{y+1} = x$

Not one-to-one (no inverse) $\quad \rightarrow \pm\sqrt{x+1} = y$ Not a function of x.