1.5 Limits

Learning outcomes:

- Find limits of functions graphically and numerically.
- Use properties of limits to evaluate limits of functions.
- Use different analytic techniques to evaluate limits of functions.
- Evaluate one-sided limits.
- Recognize unbounded behavior of functions.

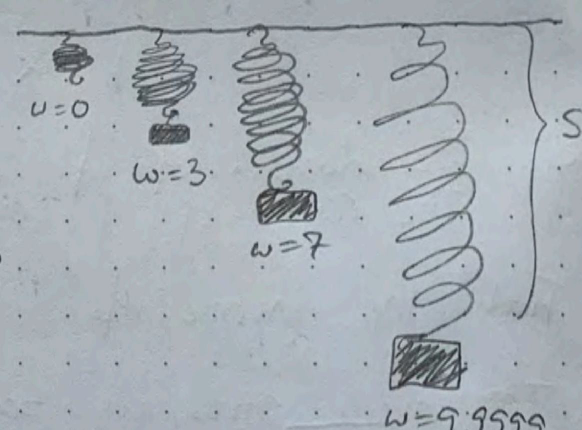
> The Limit of a Function

Consider:

- speed limit
- weight / load limit
- limit of your endurance

$\Rightarrow$ limits are <u>bounds</u>.

Consider a spring which will break if 10lbs or more is attached to it.



The spring has some length s for each weight w. At weight ≥ 10, the spring reaches length L and breaks.

We say that "the limit of s as w approaches 10 is L".

The notation for a limit is

$$\lim_{x \to c} f(x) = L$$

"the limit of f(x) as x approaches c is L"

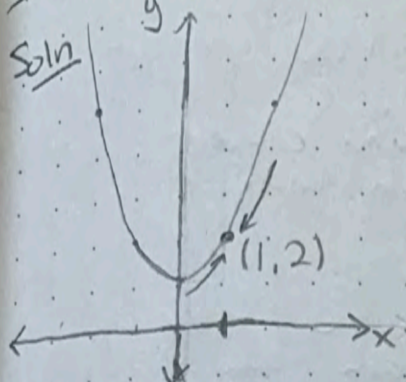
In the spring example,

$$\lim_{w \to 10} S(w) = L$$

spring length as a function of weight

ex Find the limit $\lim_{x \to 1} (x^2+1)$

Soln

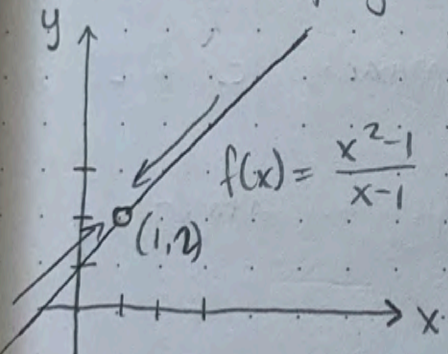


x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.81	1.98	1.998	2	2.002	2.02	2.21

Why not just plug in 1? When we use limits in practice,
- this is generally not possible!
- or there is something that makes this problematic

Ex Find $\lim_{x \to 1} \frac{x^2-1}{x-1}$

Soln Can't plug in 1! Sketch and table:



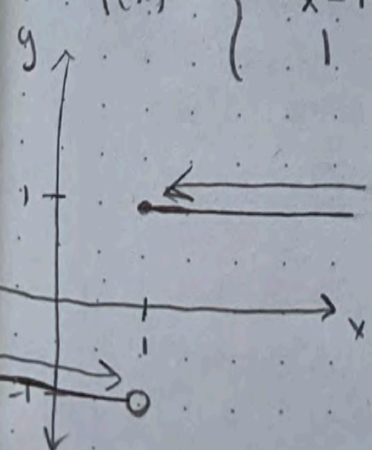
x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	???	2.001	2.01	2.1

So $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

Use a O to show a point that is NOT included on the graph.

ex
$$f(x) = \begin{cases} \frac{|x-1|}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$

Find lim as $x \to 1$.
If we plug in 1, we get 1. Is this the limit?



x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	−1	−1	−1	1	1	1	1

We get a different value from the left than from the right! In this case the limit does not exist (DNE)!

Important ideas:

1) $\lim_{x\to c} f(x) = L$ means that the value of $f(x)$ can get arbitrarily close to L as x gets closer and closer to c — $f(x)$ may or may not be such that $f(c)$ exists or $f(c) = L$.

2) For a limit to exist, x must approach c from either side and both sides must approach the same value of $f(x)$.

3) The value $f(c)$ does not necessarily tell us anything about L.

Def The limit of a function:
If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then

$$\lim_{x\to c} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches c is L".

Some Properties of Limits:
Let b, c be real numbers and let n be a positive integer.

1) $\lim_{x\to c} b = b$

2) $\lim_{x\to c} x = c$

3) $\lim_{x\to c} x^n = c^n$

4) If n is odd, $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

If n is even, $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$, $c>0$.

Operations with Limits:
let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1) Scalar multiple: $\lim_{x \to c} \left(bf(x)\right) = bL$

2) Sum or difference: $\lim_{x \to c} \left[f(x) \pm g(x)\right] = L \pm K$

3) Product: $\lim_{x \to c} \left[f(x) g(x)\right] = LK$

4) Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ provided $K \neq 0$.

5) Power: $\lim_{x \to c} \left[f(x)\right]^n = L^n$

6) Radical $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$

if n is even, then L must be positive.

Ex Find $\lim_{x \to 2} (x^2 + 2x - 3)$

Soln $\lim_{x \to 2} (x^2 + 2x - 3) = \lim_{x \to 2} (x^2) + \lim_{x \to 2} (2x) - \lim_{x \to 2} (3)$

$$= 2^2 + 2(2) - 3$$
$$= 5$$

Another property:
If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c)$$

The Replacement Theorem

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$. If the limit of g exists as $x \to c$, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

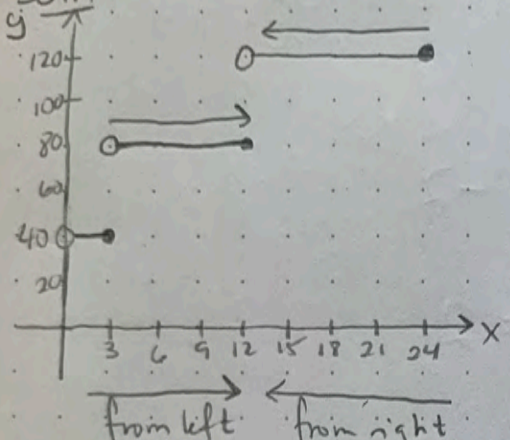
Finding such a function $g(x)$ can make our lives a lot easier!

Ex Dosage: The rec. dosage of a children's pain reliever is 40mg for infants 0 to 3 months, 8 mg for more than 3 up to 12 months, and 120 for more than 12 up thru 24 months. If x = age, we can represent this as

$$f(x) = \begin{cases} 40 & 0 < x \leq 3 \\ 80 & 3 < x \leq 12 \\ 120 & 12 < x \leq 24 \end{cases}$$

Show that the limit of $f(x)$ as $x \to 12$ DNE.

Soln



$$\lim_{x \to 12^-} f(x) = 80$$

$$\lim_{x \to 12^+} f(x) = 120$$

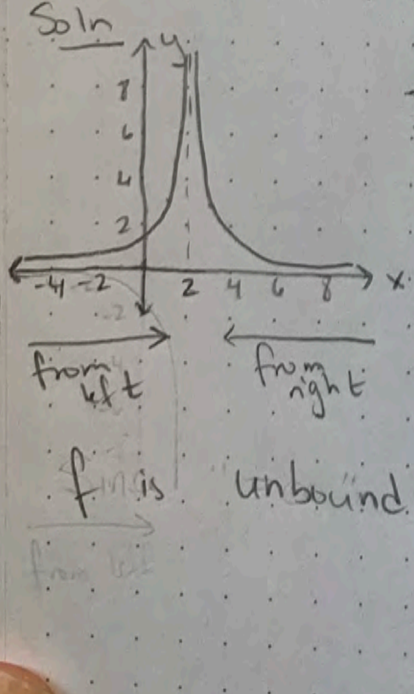
Since $80 \neq 120$, the limit ax $x \to 12$ DNE.

> Unbounded Behavior

A limit can also fail to exist if $f(x)$ increases or decreases without bound as $x \to c$.

Ex Find $\lim_{x \to 2} \left| \frac{3}{x-2} \right|$

Soln



Consider

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	30	300	3000	???	3000	300	30

$$\lim_{x \to 2^-} \left| \frac{3}{x-2} \right| = \infty$$

and

$$\lim_{x \to 2^+} \left| \frac{3}{x-2} \right| = \infty$$

f is unbounded as $x \to 2$, so the limit DNE!

One-Sided limits

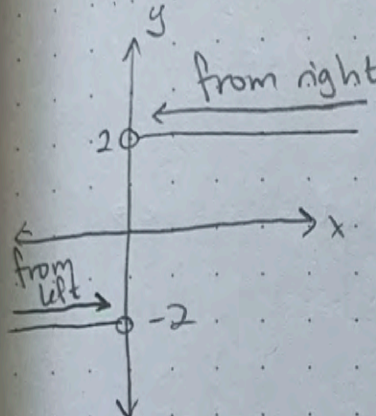
> A limit can fail to exist if a function approaches a different value from the left than it does from the right.

We can describe this behavior using one-sided limits.

$\lim_{x \to c^-} f(x) = L$ — limit from the left

$\lim_{x \to c^+} f(x) = L$ — limit from the right

Ex Given $f(x) = \frac{|2x|}{x}$ find $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$



Based on the graph,

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2$$

and

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2$$

Existence of a Limit:

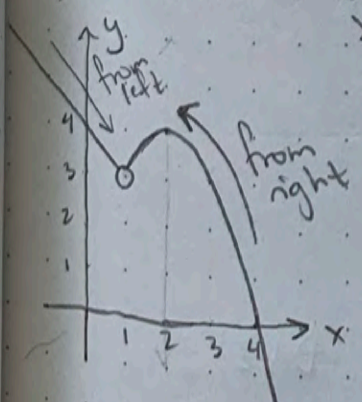
If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

iff

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

Ex Find $\lim_{x \to 1} f(x) = \begin{cases} 4-x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$



$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4-x) = 4-1 = 3$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x - x^2 = 4 - 1 = 3$$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$,

$$\lim_{x \to 1} f(x) = 3.$$

Ex Find $\lim_{x \to 1} \frac{x^3-1}{x-1}$

Soln Notice that x^3-1 and $x-1$ are both zero at $x=1$. So $x-1$ must be a factor of both.

$$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$$
$$= x^2+x+1 \quad \text{for } x \neq 1$$

Then we can apply Replacement Theorem since $(x^3-1)/(x-1)$ and x^2+x+1 agree for all $x \neq 1$.

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x \to 1} (x^2+x+1)$$
$$= 1^2+1+1$$
$$= 3$$

Ex Find $\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$

Soln Clearly we can't divide by zero, so we need to rewrite this.

Rationalize numerator:
$$\frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{x+1}-1}{x}\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$
$$= \frac{(x+1)-1}{x(\sqrt{x+1}+1)}$$
$$= \frac{x}{x(\sqrt{x+1}+1)}$$
$$= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0$$

Using the replacement theorem,
$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1}$$
$$= \frac{\lim_{x \to 0} 1}{\lim_{x \to 0} (\sqrt{x+1}+1)}$$ as long as denominator nonzero
$$= \frac{1}{\lim_{x \to 0} (\sqrt{x+1}+1)}$$
$$= \frac{1}{\sqrt{0+1}+1} = \frac{1}{1+1}$$
$$= \frac{1}{2}$$