The Product Rule

The derivative of the product of two differentiable functions f and g is

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f(x)\,g'(x) + g(x)\,f'(x)$$

Ex Find the derivative of $y = (3x - 2x^2)(5 + 4x)$

Soln Let $f(x) = 3x - 2x^2$ and $g(x) = 5 + 4x$

$f'(x) = 3 - 4x$

$g'(x) = 4$

So
$$\begin{aligned} y' &= f(x)g'(x) + g(x)f'(x) \\ &= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x) \\ &= 12x - 8x^2 + 15 - 8x - 16x^2 \\ &= 15 + 4x - 24x^2 \end{aligned}$$

Proof of the Product Rule

Let $h(x) = f(x)\,g(x)$ with f, g differentiable wrt x.

Then
$$\begin{aligned} h'(x) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} + \frac{f(x+\Delta x)g(x)}{\Delta x} - \frac{f(x+\Delta x)g(x)}{\Delta x}\right] \\ &= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \\ &= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + \lim_{\Delta x \to 0}\left[g(x)\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \\ &= \lim_{\Delta x \to 0} f(x+\Delta x) \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x) \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= f(x)\,g'(x) + g(x)\,f'(x) \end{aligned}$$

$\boxtimes$

Ex Find the derivative of $f(x) = \left(\frac{1}{x} + 1\right)(x-1)$

Soln $f'(x) = (x^{-1} + 1)(1) + (x-1)(-1x^{-2})$

$= x^{-1} + 1 - (x^{-1} - x^{-2})$

$= 1 - x^{-2}$

$= 1 - \frac{1}{x^2}$

Ex Find the derivative of (a) $y = 2x(x^2 + 3x)$ (b) $2(x^2 + 3x)$

a) $y' = 2x \frac{d}{dx}[x^2 + 3x] + (x^2 + 3x)\frac{d}{dx}[2x]$

$= 2x(2x+3) + (x^2 + 3x)(2)$

$= 4x^2 + 6x + 2x^2 + 6x$

$= 6x^2 + 12x$

b) $y' = 2\frac{d}{dx}[x^2 + 3x]$

$= 2(2x+3)$

$= 4x + 6$

Extending the Product Rule for f, g, h differentiable wrt x,

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

The same pattern holds for >3 factors as well.

The Quotient Rule

The derivative of the quotient of two differentiable functions f and g is

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Ex Find the derivative of $y = (x-1)/(2x+3)$

Soln Let $f(x) = x-1$ and $g(x) = 2x+3$

$f'(x) = 1$ $\qquad g'(x) = 2$

Then
$$y' = \frac{(2x+3)(1) - (x-1)(2)}{(2x+3)^2}$$
$$= \frac{2x+3-2x+2}{(2x+3)^2}$$
$$= \frac{5}{(2x+3)^2}$$

Proof of the Quotient Rule

Let $h(x) = f(x)/g(x)$, $g(x) \neq 0$, f, g differentiable wrt x

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)/g(x+\Delta x) - f(x)/g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g(x) f(x+\Delta x) - g(x+\Delta x) f(x)}{\Delta x \, g(x) \, g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{g(x) f(x+\Delta x) - f(x) g(x) + f(x) g(x) - f(x) g(x+\Delta x)}{\Delta x \, g(x) g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x \, g(x) g(x+\Delta x)} \mp \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x \, g(x) \, x + \Delta x}$$

$$= \frac{g(x) \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} - f(x) \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}}{g(x) \lim_{\Delta x \to 0} g(x+\Delta x)}$$

$$= \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$$

∎

Ex Find the equation of the tangent line to $y = \dfrac{2x^2-4x+3}{2-3x}$ when $x=1$.

Soln
$$\frac{dy}{dx} = \frac{(2-3x)\frac{d}{dx}[2x^2-4x+3] - (2x^2-4x+3)\frac{d}{dx}[2-3x]}{(2-3x)^2}$$

$$= \frac{(2-3x)(4x-4) - (2x^2-4x+3)(-3)}{(2-3x)^2}$$

$$= \frac{-12x^2+20x-8 - (-6x^2+12x-9)}{(2-3x)^2}$$

$$= \frac{-6x^2+8x+1}{(2-3x)^2}$$

At $x=1$, $y' = \dfrac{-6+8+1}{(2-3)^2} = 3$

$$y = \frac{2-4+3}{2-3} = -1$$

So the tangent line is

$y - y_0 = m(x-x_0)$

$y+1 = 3(x-1)$

$y+1 = 3x-3$

$y = 3x-4$

Simplifying derivatives

Rewriting before differentiating

original	rewrite	differentiate	Simplify
$y = \dfrac{x^2+3x}{6}$	$y = \frac{1}{6}(x^2+3x)$	$y' = \frac{1}{6}(2x+3)$	$y' = \frac{1}{3}x + \frac{1}{2}$
$y = \dfrac{5x^4}{8}$	$y = \frac{5}{8}x^4$	$y' = \frac{5}{8}(4x^3)$	$y' = \frac{5}{2}x^3$
$y = \dfrac{-3(3x-2x^2)}{7x}$	$y = -\frac{3}{7}(3-2x)$	$y' = -\frac{3}{7}(-2)$	$y' = \frac{6}{7}$
$y = \dfrac{9}{5x^2}$	$y = \frac{9}{5}x^{-2}$	$y' = \frac{9}{5}(-2x^{-3})$	$y' = -\frac{18}{5}x^{-3}$

We can combine the product and quotient rules:

Ex Find the derivative of $y = \frac{(1-2x)(3x+2)}{5x-4}$

$$y' = \frac{(5x-4)\frac{d}{dx}[(1-2x)(3x+2)] - (1-2x)(3x+2)\frac{d}{dx}[5x-4]}{(5x-4)^2}$$

$$= \frac{5x-4\left[(1-2x)\frac{d}{dx}[3x+2] + (3x+2)\frac{d}{dx}[1-2x]\right] - (1-2x)(3x+2)\frac{d}{dx}[5x-4]}{(5x-4)^2}$$

$$= \frac{(5x-4)[(1-2x)(3) + (3x+2)(-2)] - (1-2x)(3x+2)(5)}{(5x-4)^2}$$

$$= \frac{(5x-4)(-12x-1) - (1-2x)(15x+10)}{(5x-4)^2}$$

$$= \frac{-60x^2 + 43x + 4 - (-30x^2 - 5x + 10)}{(5x-4)^2}$$

$$= \frac{-30x^2 + 48x - 6}{(5x-4)^2}$$

Application: Rate of Change of Systolic Blood Pressure

Consider a person whose systolic blood pressure P (mm Hg) is given by

$$P = \frac{25t^2 + 125}{t^2 + 1} \qquad 0 \le t \le 10$$

where t is time in seconds. At what rate is the blood pressure changing 5 seconds after leaving the heart?

Soln

$$\frac{dP}{dt} = \frac{(t^2+1)(50t) - (25t^2+125)(2t)}{(t^2+1)^2}$$

$$= \frac{50t^3 + 50t - 50t - 250t}{(t^2+1)^2}$$

$$= \frac{-200t}{(t^2+1)^2}$$

At $t=5$, $= \frac{-200(5)}{(25+1)^2} = -1.48$ mm Hg/sec

The pressure drops by 1.48 mm Hg/second at 5 sec after leaving the heart.