Math 26A - Exam 2 Key

Drinking Horn

$$0 < r \le \frac{4}{\sqrt{\pi}}$$
$$0 < h < \frac{16}{\pi}$$

$$A = \pi r \sqrt{h^2 + r^2}$$ (remove "lid")

$$V = \pi r^2 \frac{h}{3} = 16$$ (don't worry about the units!)

$$\Rightarrow h = \frac{48}{\pi r^2}$$

So
$$A = \pi r \sqrt{\frac{2304}{\pi^2} r^{-4} + r^2}$$

rewrite to make derivative easier

$r > 0$

$$= \left(\pi^2 r^2\right)^{1/2} \left(\frac{2304}{\pi^2} r^{-4} + r^2\right)^{1/2}$$

$$= \left(\frac{\pi^2 r^2 (2304)}{\pi^2 r^4} + \pi^2 r^2 r^2\right)^{1/2}$$

$$= \left(\frac{2304}{r^2} + \pi^2 r^4\right)^{1/2}$$

$$= \left(\frac{2304 + \pi^2 r^6}{r^2}\right)^{1/2}$$

$$= \frac{(2304 + \pi^2 r^6)^{1/2}}{r}$$

$u = (2304 + \pi^2 r^6)^{1/2}$

$v = r \qquad dv = 1$

$du = \frac{1}{2}(2304 + \pi^2 r^6)^{-1/2}(6\pi^2 r^5)$

$$\frac{dA}{dr} = \frac{\frac{6}{2} r (2304 + \pi^2 r^6)^{-1/2}(\pi^2 r^5) - (2304 + \pi^2 r^6)^{1/2}}{r^2}$$

$$= \frac{3r^6(\pi^2)}{r^2\sqrt{2304 + \pi^2 r^6}} - \frac{\sqrt{2304 + \pi^2 r^6}}{r^2}$$

$$= \frac{3\pi^2 r^6 - \pi^2 r^6 - 2304}{r^2\sqrt{2304 + \pi^2 r^6}}$$

$$= \frac{2\pi^2 r^6 - 2304}{r^2\sqrt{2304 + \pi^2 r^6}}$$

This is never divided by zero $(r > 0)$ and is zero when numerator $= 0$

So
$$0 = 2\pi^2 r^6 - 2304$$

$$r^6 = \frac{2304}{2\pi^2} \Rightarrow r \approx 2.211 \text{ so } h = \frac{48}{\pi r^2} \approx 3.432$$

Function: $h(x) = \dfrac{2x^2+7}{1.3-x}$

$u = 2x^2+7$ $\quad du = 4x$

$v = 1.3 - x$ $\quad dv = -1$

$$h'(x) = \frac{(1.3-x)(4x)-(2x^2+7)(-1)}{(1.3-x)^2}$$

$$= \frac{5.2x - 4x^2 + 2x^2 + 7}{(1.3-x)^2}$$

$$= \frac{-2x^2+5.2x+7}{(1.3-x)^2}$$

undefined at $x = 1.3$

$$0 = -2x^2+5.2x+7$$

$$x = \frac{-5.2 \pm \sqrt{5.2^2-4(-2)(7)}}{2(-2)}$$

$x \approx -0.9782$ and $x \approx 3.5782$

1) Relative extrema:

Interval	Sign of $h'(x)$	Conclusion
$(-\infty, -0.9782)$	$-$	decreasing
$(-0.9782, 1.3)$	$+$	increasing
$(1.3, 3.5782)$	$+$	increasing
$(3.5782, \infty)$	$-$	decreasing

$\left.\begin{matrix}\text{decreasing}\\ \text{increasing}\end{matrix}\right\}$ relative minimum

$\left.\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}\right\}$ relative maximum

There are relative extrema at $(-0.9782, 3.913)$ (minimum) and at $(3.578, -14.313)$ (max.)

$$\frac{2(-0.9782)^2+7}{1.3+0.9782} = 3.913$$

$$\frac{2(3.5782)^2+7}{1.3-3.5782} = -14.313$$

2) Absolute extrema on $[-4, 1]$

Value	$h(x)$	
-4	7.358	
-0.9782	3.913	$\rightarrow$ minimum
1	30	$\rightarrow$ maximum

$$\frac{2(-4)^2+7}{1.3+4} = 7.358$$

$$\frac{2(1)^2+7}{1.3-1} = 30$$

The absolute min is at $(-0.9782, 3.913)$ and the abs. max. is $(1, 30)$

3) Asymptotes:

$1.3 - x = 0$ when $x = 1.3$, so this is a vertical asymptote

$$h(x) = \frac{2x^2+7}{1.3-x} = \frac{2x^2+7}{-x+1.3} \quad \frac{\text{degree } 2}{\text{degree } 1}$$

Since degree(numerator) > degree(denominator), there are no horizontal asymptotes.

Weibull distribution

$$f(x) = \begin{cases} 1 - e^{-\left(\frac{x}{\lambda}\right)^k} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

1) $y = 1 - e^{-\left(\frac{x}{\lambda}\right)^k}$

$$e^{-\left(\frac{x}{\lambda}\right)^k} = 1 - y$$

$$-\left(\frac{x}{\lambda}\right)^k = \ln(1-y)$$

$$\left(\frac{x}{\lambda}\right)^k = -\ln(1-y)$$

$$\frac{x}{\lambda} = \left[-\ln(1-y)\right]^{1/k}$$

$$\boxed{x = \lambda \sqrt[k]{-\ln(1-y)}}$$

2) $f(x)$ for $k=3$, $\lambda = 7$, $x = 10$

$$f(x) = 1 - e^{-\left(\frac{10}{7}\right)^3}$$

$$= 1 - e^{-2.915}$$

$$\approx 1 - 0.0542$$

$$\boxed{\approx 0.9458}$$

3. Probability density function

$f'(x) = 0$ for $x < 0$

$$f'(x) = -\left[e^{-\left(\frac{x}{\lambda}\right)^k} \frac{d}{dx}\left[-\left(\frac{x}{\lambda}\right)^k\right]\right]$$

$$= -e^{-\left(\frac{x}{\lambda}\right)^k} \frac{d}{dx}\left[-\frac{x^k}{\lambda^k}\right]$$

$$= \frac{1}{\lambda^k} e^{-\left(\frac{x}{\lambda}\right)^k} \frac{d}{dx}\left[x^k\right]$$

$$= \frac{k}{\lambda^k} e^{-\left(\frac{x}{\lambda}\right)^k} x^{k-1} \quad \text{for } x \geq 0$$

$$\boxed{= \frac{kx^{k-1}}{\lambda^k} e^{-\left(\frac{x}{\lambda}\right)^k}}$$